

# Non-Markovianity as a function of system size

Felipe F. Fanchini,<sup>1</sup> Göktuğ Karpat,<sup>2</sup> Leonardo K. Castelano,<sup>3</sup> and Daniel Z. Rossatto<sup>3</sup>

<sup>1</sup>*Faculdade de Ciências, UNESP - Universidade Estadual Paulista, Bauru, SP, 17033-360, Brazil*

<sup>2</sup>*Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla, Istanbul, 34956, Turkey*

<sup>3</sup>*Departamento de Física, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil*

(Dated: March 4, 2013)

We study the non-Markovianity of the dynamics of open quantum systems as a function of system size. We investigate two distinct quantum processes, namely dephasing and relaxation, by considering independent and common environments. For the zero temperature case, we present an efficient method to evaluate a measure of non-Markovianity recently proposed by Luo, Fu, and Song [S. Luo, S. Fu, and H. Song, Phys. Rev. A 86, 044101 (2012)]. We perform a comprehensive analysis of non-Markovianity for the case of a single qubit, including the determination of the optimal initial states, that is required for the calculation of this new measure, as a function of the parameters of the environment. We demonstrate that, for independent environments, the degree of non-Markovianity for dephasing and relaxation processes can increase with the number of the qubits in the system. Furthermore, we discuss the same processes for two qubits interacting with a common environment. In this case, we show that while the amount of non-Markovianity increases as compared to the single qubit case when a common relaxation process is considered, our analysis suggests that the same is not generally true for the dephasing process.

PACS numbers: 03.65.Yz, 42.50.Lc

## I. INTRODUCTION

The concept of non-Markovianity is a prominent aspect of the dynamics of open quantum systems that has been attracting both theoretical and experimental attention [1]. It has been shown that non-Markovianity can be used as a tool in quantum protocols [2], can be employed to take advantage in quantum metrology [3], and can be exploited in quantum key distribution [4]. Other concepts behind the non-Markovian dynamics have also been investigated, for example, the influence of environment size [5] and the possibility to pursue new quantum technologies by using non-Markovian effects [2]. Although all those efforts have been performed, measuring non-Markovianity is complicated and generally very small systems have yet been considered [6]. However, the real usefulness of a quantum system for computation or simulation is only appreciable in the limit of large-scale information processing. Therefore, it is critical to understand the scaling properties of non-Markovianity for multipartite systems. To the best of our knowledge, this is the first study concerning the scaling problem for a non-Markovianity measure.

Recently, various measures for quantifying the degree of non-Markovianity of the dynamics of an open quantum system [7–10] have been introduced in the literature; however, there is no consensus on what precisely determines the non-Markovianity of a dynamical quantum process. It has been demonstrated that the conclusions drawn from different measures might not agree depending on the considered physical model. One of the most widely used measures of non-Markovianity has been introduced by Breuer, Laine and Piilo (BLP) [7]. In their seminal paper, they claimed that information flows only from the system into the environment for a Markovian process and the information flow can be measured by the trace distance of two arbitrary quantum states, which probes the distinguishability between them. To implement this measure, one needs to perform an optimization by

checking the dynamics of the trace distance for a huge number of initial states. Thus, this procedure is very demanding and almost impracticable when dealing with multipartite systems. Moreover, Rivas, Huelga and Plenio (RHP) have constructed a measure of non-Markovianity that quantifies the deviation from divisibility for a dynamical map [8], which is also hard to be implemented in general. In fact, the evaluation of almost all known non-Markovianity measures is very complicated for multipartite systems. In this work, in order to overcome this difficulty, we present an efficient method to evaluate a measure of non-Markovianity, recently proposed by Luo, Fu, and Song (LFS), based on the non-monotonical behavior of the quantum mutual information for non-Markovian processes [9]. This measure coincides with other important ones such as BLP and RHP measures for quite general cases and can be straightforwardly extended for studying multipartite systems.

When a quantum system interacts with its environment, there are two important physical processes that must be considered: relaxation and decoherence (here called dephasing). While relaxation is associated to a process involving loss of energy, dephasing is associated to the loss of purity without any energy exchange between the system and its surroundings. Indeed, the density matrix population is constant in time for the latter process. In this work, we explore both processes considering independent and common phase damping (PD) and amplitude damping (AD) channels to describe dephasing and relaxation, respectively. For a zero temperature environment, we link the LFS measure to the rate of change of the system entropy  $S(\rho^s(t))$  and the environment entropy  $S(\rho^e(t))$ . In fact, we show that a quantum process is non-Markovian if the time derivative of  $S(\rho^s(t))$  is greater than the time derivative of  $S(\rho^e(t))$  at any instant of time. We present a detailed analysis of the evaluation of the LFS measure for a single qubit, and discuss the behavior of the optimal initial states as a function of the parameters of the environments. Moreover, we demon-

strate that the degree of non-Markovianity can *increase* as a function of the number of qubits for the cases of independent PD and AD channels. We also investigate the non-Markovian behavior of global PD and AD channels for a system of two qubits. Lastly, we compare the results obtained for the LFS measure and its simplified version, whose evaluation does not include a complex optimization procedure.

## II. MEASURING NON-MARKOVIANITY

The definition of the LFS measure of non-Markovianity [9] is based on the following: suppose that we have a quantum system in a Hilbert space  $H$ , and a quantum process  $\Lambda(t)$  governing the dynamical evolution of the considered system. If an arbitrary ancilla system in a Hilbert space  $H^a$  is introduced, the composite state of the principal system and ancilla  $\rho^{sa}$  pertains to the Hilbert space  $H \otimes H^a$ . In this case, assuming a trivial dynamics on the ancillary, the time evolution of the total system is given by  $\rho^{sa}(t) = (\Lambda(t) \otimes I)\rho^{sa}(0)$ , where  $I$  is the identity operator acting on the state space of the ancillary. The amount of total correlations in a bipartite state  $\rho^{sa}$  can be quantified through the quantum mutual information

$$I(\rho^{sa}) = S(\rho^s) + S(\rho^a) - S(\rho^{sa}), \quad (1)$$

where  $\rho^a = \text{tr}_s \rho^{sa}$  and  $\rho^s = \text{tr}_a \rho^{sa}$  represent the reduced density operators of the system and the ancilla, respectively.  $S(\rho) = -\text{tr} \rho \log_2 \rho$  is the von Neumann entropy. Exploiting the fact that quantum mutual information is a monotonically decreasing function of time for a Markovian process, LFS have proposed a new quantity for measuring the non-Markovianity of the dynamical process  $\Lambda(t)$  from an information theoretical perspective:

$$\mathcal{N}(\Lambda) = \sup_{\rho^{sa}(0)} \int_{(d/dt)I(\rho^{sa}(t)) > 0} \frac{d}{dt} I(\rho^{sa}(t)) dt, \quad (2)$$

where the sup is taken over all possible initial states  $\rho^{sa}(0)$ . Even though this measure has a significant meaning for the quantification of non-Markovianity, its evaluation is hard to perform due to the potentially complex optimization problem.

As described in Eq. (2), a dynamical quantum process is said to be non-Markovian if  $\frac{d}{dt} I(\rho^{sa}(t)) > 0$  at any instant of time. Note that the ancilla, unlike the system, does not interact with the environment. In other words, the state of the ancilla is time independent. Therefore, the time derivative of the quantum mutual information can be written as

$$\frac{d}{dt} I(\rho^{sa}(t)) = \frac{d}{dt} S(\rho^s(t)) - \frac{d}{dt} S(\rho^a(t)). \quad (3)$$

For a zero temperature environment, an interesting result can be obtained from this equation. Since we take  $\rho^{sa}(0)$  as a pure state, and the environment starts in the state  $\rho^e(0) = |0\rangle\langle 0|$  for the zero temperature case, the total quantum state composed of the system, the ancilla and the environment is a pure state at any time, leading to  $S(\rho^{sa}(t)) = S(\rho^e(t))$ . Consequently, following the LFS measure, we obtain a non-Markovianity

criterion without the need of an ancilla: a quantum process is non-Markovian if and only if

$$\frac{d}{dt} S(\rho^s(t)) > \frac{d}{dt} S(\rho^e(t)). \quad (4)$$

This condition links the non-Markovianity measure to the rate of change of the system and the environment entropies. Nonetheless, some peculiar aspects should be noted: the environment is initially in a pure state by assumption, but no restrictions were imposed to the system. If the system is initially in a pure state as well, we have  $\rho^{se}$  pure and therefore  $S(\rho^s(t)) = S(\rho^e(t))$ . Such a result does not mean that the process is actually Markovian because we need to maximize over all possible initial conditions of the system to be able to determine the degree of non-Markovianity. An equivalent equation for Eq. (2) can be deduced without the necessity of an ancilla:

$$\mathcal{N}_{T=0K}(\Lambda) = \sup_{\rho^s(0)} \int_{(d/dt)\Delta S_{se}(t) > 0} \frac{d}{dt} \Delta S_{se}(t) dt, \quad (5)$$

with  $\Delta S_{se}(t) = S(\rho^s(t)) - S(\rho^e(t))$ . The advantage of the above equation over Eq. (2) is that the maximization is just over the initial conditions of the system instead of the initial conditions of the composite state of the system and the ancilla, that is required to calculate  $\mathcal{N}(\Lambda)$ . It is important to emphasize that, for a non-Markovian process, even though the entropies of system and environment can increase or decrease, the main point is connected to the time derivatives of the entropies. Furthermore, to calculate  $S(\rho^e(t))$ , we do not need to worry about the state of the environment. The idea is to maximize  $\mathcal{N}_{T=0K}$  over all possible initial system states and, for each choice, purify it including an extra subsystem. Because the environment is set in a pure state at  $t = 0$ , the entropy of the system plus the purifier subsystem is equal to the entropy of the environment at any time.

On the other hand, LFS presented a significant simplification for Eq. (2) in ref. [9]. Assuming that  $H^a = H$  and  $\rho^{sa}(0) = |\Psi\rangle\langle\Psi|$  where  $|\Psi\rangle$  is *any* maximally entangled pure state of the system and the ancilla, they obtain an easily computable measure of non-Markovianity:

$$\mathcal{N}_0(\Lambda) = \int_{(d/dt)I(\rho^{sa}(t)) > 0} \frac{d}{dt} I(\rho^{sa}(t)) dt, \quad (6)$$

with  $\rho^{sa}(t) = (\Lambda(t) \otimes I)|\Psi\rangle\langle\Psi|$ . In this work, we explicitly show that  $\mathcal{N}_0(\Lambda)$ , despite its utility as a witness for non-Markovianity, may be misleading and give an inaccurate conclusion about the degree of non-Markovianity of a quantum process. Furthermore, we demonstrate that  $\mathcal{N}(\Lambda)$  does not depend on the amount of entanglement shared between the system and ancilla because two distinct initial states with the same degree of entanglement can give different results. Actually, the optimal state is not maximally entangled in general.

### III. OPEN SYSTEM DYNAMICS

#### A. Phase Damping Channel

We consider a spin-boson type Hamiltonian  $H_{IPD}$  that describes the interaction between a qubit and a bosonic environment:

$$H_{IPD} = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k a_k^\dagger a_k + \sum_k \sigma_z (g_k a_k^\dagger + g_k^* a_k), \quad (7)$$

where the first and the second terms are responsible for the free evolution of the qubit and the environment, respectively. The third term accounts for the interaction between the qubit and its environment. We first note that  $[H, \sigma_z] = 0$ , which immediately implies the absence of transitions between different energy levels. Thus, the population terms in the density matrix of the system are conserved quantities. Here,  $\omega_0$  is the transition frequency of the qubit and  $\omega_k$  is the field frequency of the  $k$ -th environmental field mode. The constant  $g_k$  controls the strength of the coupling between the qubit and each field mode of the environment. While the qubit operator is given by the usual Pauli  $\sigma_z$  matrix, the creation operator  $a_k$  and the annihilation operator  $a_k^\dagger$ , satisfying the bosonic commutation relations  $[a_k, a_{k'}^\dagger] = \delta_{k,k'}$ , represent the environment. It is worth to stress that this qubit plus environment model admits an exact solution. We assume that the composite state of the qubit and the environment are initially factorized, that is, there exists no correlations between the system and the environment at  $t = 0$ ; furthermore, the environment is initially in its vacuum state  $\rho^e(0) = |0\rangle\langle 0|$  at zero temperature. We consider a sufficiently large environment for which we can replace the sum over the discrete coupling constants with an integral over a continuous distribution of frequencies of the environmental modes, *i.e.*,  $\sum_k |g_k|^2 \rightarrow \int_0^\infty d\omega J(\omega)$ . In addition, we suppose that the spectral density of the environmental modes is Ohmic-like, that is

$$J(\omega) = \eta \frac{\omega^s}{\omega_c^{s-1}} e^{-\omega/\omega_c}, \quad (8)$$

with  $\omega_c$  being the cut-off frequency and  $\eta$  a dimensionless coupling constant. Depending on the parameter  $s$ , the spectral density is called subohmic ( $s < 1$ ), ohmic ( $s = 1$ ) or superohmic ( $s > 1$ ). Under these conditions, the dynamics of a single qubit can be obtained in the operator-sum representation as

$$\rho(t) = \sum_{i=1}^2 K_i(t) \rho(0) K_i^\dagger(t), \quad (9)$$

where the Kraus operators  $K_i(t)$  are given by

$$K_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & r(t) \end{pmatrix}, \quad K_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - r^2(t)} \end{pmatrix}, \quad (10)$$

with  $\sum_{i=1}^2 K_i^\dagger(t) K_i(t) = I$  for all values of  $t$ , where  $I$  denotes the  $2 \times 2$  identity matrix. Here, the dephasing parameter

$r(t)$  is

$$r(t) = \exp \left[ - \int_0^t \gamma(t') dt' \right], \quad (11)$$

where the dephasing rate takes the form

$$\gamma(t) = \eta \omega_c (1 + (\omega_c t)^2)^{-s/2} \Gamma(s) \sin(s \arctan(\omega_c t)), \quad (12)$$

with  $\Gamma(s)$  being the Euler gamma function.

#### B. Amplitude Damping Channel

In order to discuss the relaxation process, we consider the following model Hamiltonian  $H_{IAD}$

$$H_{IAD} = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k^\dagger a_k + (\sigma_+ B + \sigma_- B^\dagger), \quad (13)$$

where  $B = \sum_k g_k a_k$  with  $g_k$  being the coupling constant. The first two terms of Eq.(13) describe the free evolution of the qubit and the environment, respectively, while the third term accounts for the interaction between the qubit and the environment. The transition frequency of the qubit is  $\omega_0$ , and  $\sigma_\pm$  denotes the raising and lowering operators of the qubit. The index  $k$  is used to label the different environmental field modes with frequencies  $\omega_k$ , which are mathematically described by the annihilation and creation operators given by  $a_k$  and  $a_k^\dagger$ , respectively. Restricting ourselves to the case of a single excitation, the modes of the environment can be described by an effective spectral density of the form

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2}, \quad (14)$$

where  $\lambda$  defines the spectral width of the coupling and is also connected to the correlation time of the environment  $\tau_B$  by the relation  $\tau_B \approx 1/\lambda$ .  $\gamma_0$  is the time scale  $\tau_R$  over which the state of the system changes by  $\tau_R \approx 1/\gamma_0$ . For this form of a spectral density, it is not hard to distinguish the weak and the strong coupling regimes. The case  $\tau_R > 2\tau_B$  corresponds to the weak coupling regime where the decoherence process is Markovian since the relaxation time is greater than the correlation time of the environment. On the other hand, the case  $\tau_R < 2\tau_B$  corresponds to the strong coupling regime where the non-Markovian nature of the environment becomes evident. We note that at zero temperature this Hamiltonian with the considered spectral density (known as the damped Jaynes-Cummings model in the literature) represents one of the few exactly solvable models for open quantum systems. In the strong coupling regime, the time evolution of a single qubit can be expressed in the operator-sum representation as

$$\rho(t) = \sum_{i=1}^2 M_i(t) \rho(0) M_i^\dagger(t), \quad (15)$$

where the corresponding Kraus operators  $M_i(t)$  are given by

$$M_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p(t)} \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 0 & \sqrt{1 - p(t)} \\ 0 & 0 \end{pmatrix}, \quad (16)$$

satisfying the condition  $\sum_{i=1}^2 M_i^\dagger(t) M_i(t) = I$  for all values of  $t$ . The damping parameter  $p(t)$  reads

$$p(t) = e^{-\lambda t} \left[ \cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d} \sin\left(\frac{dt}{2}\right) \right]^2, \quad (17)$$

with  $d = \sqrt{2\gamma_0\lambda - \lambda^2}$ .

#### IV. NON-MARKOVIANITY FOR A SINGLE QUBIT

Before starting to elucidate the scaling properties of the LFS measure for multipartite systems at zero temperature, we consider the case of a single qubit system. In this case, the maximization in Eq. (5) can be numerically evaluated because the general form of the density matrix  $\rho^s(0)$  depends just on three real variables. Explicitly,

$$\rho^s(0) = \begin{pmatrix} \rho_{11}(0) & \Re[\rho_{12}(0)] + \Im[\rho_{12}(0)] \\ \Re[\rho_{12}(0)] - \Im[\rho_{12}(0)] & 1 - \rho_{11}(0) \end{pmatrix}.$$

For the dephasing process, our numerical analysis show that the maximum present in Eq. (2) is reached for a maximally mixed initial state. In other words, the optimal state of the composite system  $\rho^{sa}(0)$  is maximally entangled, justifying the simplification proposed by LFS, as described in Eq. (6). However, contrarily to what is expected, the maximum value in Eq. (2) for the relaxation process is obtained for a diagonal initial state whose system plus ancilla is *not* maximally entangled.

As suggested by our numerical investigation, we first take the off-diagonal elements of the density matrix to be zero, that is,  $\rho_{12}(0) = \rho_{21}(0) = 0$ . In Fig. (1-a) we plot the possible values of the degree of non-markovianity quantified by the LFS measure  $\mathcal{N}(\Lambda)$ , for the PD channel with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$  as a function of the density matrix population  $\rho_{11}(0)$ . As can be seen from the figure (1-a), the LFS measure corresponds to having  $\rho_{11}(0) = 0.5$ , implying that the optimal initial composite state is maximally entangled. On the other hand, Fig. (1-b) presents the results of the same analysis performed for the AD channel with  $\gamma_0 = 1$  and  $\lambda = 0.1$ . It is shown that the LFS measure is obtained for  $\rho_{11}(0) \approx 0.4$ , which means that the optimal composite state is not maximally entangled in this case. In particular, such a result points out that the LFS measure does not generally depend on the initial entanglement between the system and the ancilla.

Another interesting point is related with the dependence of the optimal initial state of the system on the parameters of the considered environmental model. In Fig. (2-a) and Fig. (2-b), we plot the density matrix element  $\rho_{11}(0)$  of the optimal initial state for the dephasing and the relaxation processes, respectively, as a function of the bath parameters  $\omega_c$  and  $\lambda$ . The results of this analysis demonstrate an important difference between the two cases. Whereas the optimal initial state for the PD channel does not depend on the bath parameter  $\omega_c$ , the AD channel is highly sensitive to the bath parameter  $\lambda$ . In fact, we have found that as the parameter  $\lambda$  gets smaller, the optimal state tends to be a maximally mixed one.

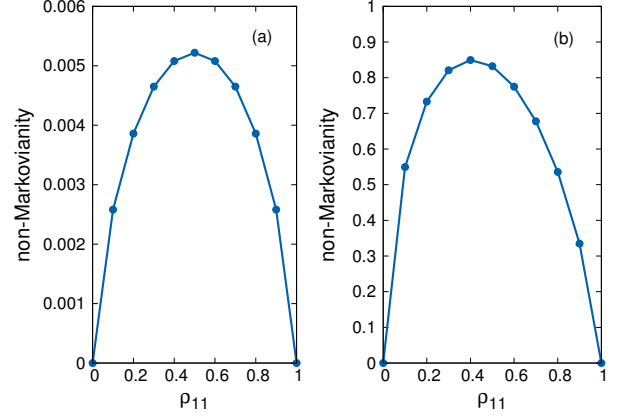


FIG. 1: Non-Markovianity for one qubit as a function of the density matrix element  $\rho_{11}(0)$  for the dephasing process (a) with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$ , and for the relaxation process (b) with  $\gamma_0 = 1$  and  $\lambda = 0.1$ .

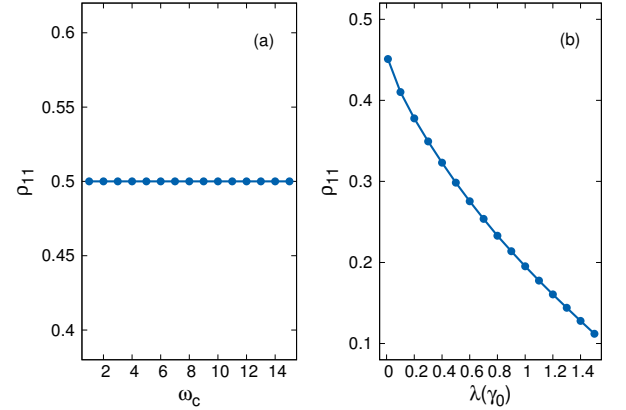


FIG. 2: The density matrix element  $\rho_{11}(0)$  of the optimal initial state of a single qubit system as a function of the bath parameters for the dephasing process (a) with  $s = 3$  and  $\eta = 2$ , and for the relaxation process (b).

#### V. SCALING OF NON-MARKOVIANITY

##### A. Independent Environments

Having collected all the required tools, we are in a position to evaluate the LFS measure for the multipartite case for independent environments. First of all, we consider a quantum state which is composed of a system of  $n$  qubits and an ancilla that purifies the system state. Assuming that only  $n$  qubits are subjected to the environmental noise, and the ancillary system evolves freely, the dynamics of the composite system  $\rho^{sa}(t)$  can be obtained as

$$\rho^{sa}(t) = \sum_{i=\{1,2\}} (E_i^{\otimes n} \otimes I) \rho^{sa}(0) (E_i^{\otimes n} \otimes I)^\dagger, \quad (18)$$



where  $E_i$  are the Kraus operators describing AD or PD channels for a single qubit,  $n$  is the number of qubits,  $I$  denotes the identity matrix with dimensions of the ancillary system, and the summation over the index  $i = \{1, 2\}$  runs over all possible permutations of the Kraus operators  $E_i^{\otimes n}$ .

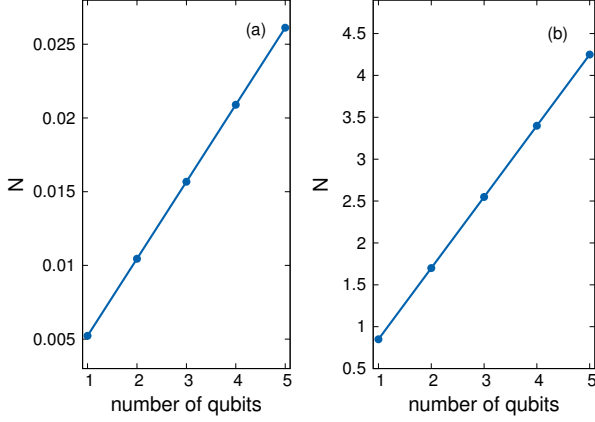


FIG. 3: Non-Markovianity  $\mathcal{N}(\Lambda)$  for the dephasing process (a) with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$ , and for the relaxation process (b) with  $\gamma_0 = 1$  and  $\lambda = 0.1$ , as a function of the number of qubits. The graphs are for independent environments.

Performing the optimization required for the calculation of the LFS measure  $\mathcal{N}(\Lambda)$  becomes a very difficult task for a system of two or more qubits due to the significantly increasing number of variables involved in these cases. To overcome this difficulty, we limit our study to diagonal product initial states of the form  $\rho^s(0)^{\otimes n}$  for multipartite systems. Indeed, this is a reasonable choice given the results acquired for the case of a single qubit and, moreover, using such diagonal states enables us to obtain a lower bound to the degree of non-Markovianity of a quantum process. While we plot the LFS measure as a function of the number of qubits for superohmic dephasing environments with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$  in Fig. (3-a), similar results are displayed for the relaxation process with  $\gamma_0 = 1$  and  $\lambda = 0.1$  in Fig. (3-b). These plots indicate a linear increase in the degree of non-Markovianity for the considered initial states, which proves that, for independent PD and AD channels, the LFS measure is *at least* additive, *i.e.*,

$$\mathcal{N}(\Lambda^{\otimes n}) \geq n[\mathcal{N}(\Lambda)]. \quad (19)$$

This peculiar behavior of the LFS measure, despite non-intuitive, is simple to be understood in this context because both the system and the environment entropies are additive quantities when independent environments are considered.

### B. Common Environment

In this section, we turn our attention to the behavior of non-Markovianity when the system is globally interacting with common dephasing and relaxation environments. Due to the

difficulty of calculating the dynamics for many qubits considering their interaction with common environments, we restrict our analysis to only two qubits. On the other hand, the analysis of the two-qubit case is still interesting to infer the main characteristics of the degree of non-Markovianity as a function of the system scale. We first consider the case where both qubits are coupled to a common dephasing bath, described by the following Hamiltonian

$$H_{CPD} = \frac{1}{2} \sum_n \omega_0^{(n)} \sigma_z^{(n)} + \sum_k \omega_k a_k^\dagger a_k + \sum_{n,k} \sigma_z^{(n)} (g_k a_k^\dagger + g_k^* a_k), \quad (20)$$

where the index  $n$  denotes the terms related to the first ( $n = 1$ ) and second ( $n = 2$ ) qubit. As a simplification, we only consider diagonal two-qubit initial states and evaluate the degree of non-Markovianity measured by the LFS measure. The results of our analysis suggest that the LFS measure  $\mathcal{N}(\Lambda)$  suffers a very significant decay as compared to the single qubit case, having a value of the order of  $10^{-7}$ .

Next, we focus on the scenario where a system of two qubits globally interact with a common relaxation environment:

$$H_{CAD} = \sum_n \omega_0^{(n)} \sigma_+^{(n)} \sigma_-^{(n)} + \sum_k \omega_k a_k^\dagger a_k + \sum_n (\sigma_+^{(n)} B + \sigma_-^{(n)} B^\dagger). \quad (21)$$

In this case our investigation reveals that, assuming diagonal initial states, the degree of non-Markovianity is significantly amplified when compared to the single qubit case, approximately turning out to be  $\mathcal{N}(\Lambda) \approx 6.21$ . Interestingly, among the two-qubit diagonal states that we have considered, the optimal one is always the maximally mixed state, independently of the bath parameters. Such a finding is rather surprising because the optimal initial state is strongly dependent on the parameters of the environment for the single qubit case.

## VI. COMPARISON OF $\mathcal{N}(\Lambda)$ AND $\mathcal{N}_0(\Lambda)$

We now compare the results obtained for the LFS measure  $\mathcal{N}(\Lambda)$  and its simplified version  $\mathcal{N}_0(\Lambda)$  for multipartite systems, considering independent environments. While the former involves a difficult maximization over all possible initial states, the latter can be directly calculated choosing a specific initial state, which we choose as a GHZ type state. Due to this restriction, it is clear that  $\mathcal{N}_0(\Lambda)$  underestimates the degree of non-Markovianity, and consequently  $\mathcal{N}_0(\Lambda) \leq \mathcal{N}(\Lambda)$ . In Fig. (4-a) we plot the logarithm of the simplified LFS measure,  $\ln(\mathcal{N}_0(\Lambda))$ , as a function of the number of qubits for the dephasing process with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$ . It can be observed that the degree of non-Markovianity measured by  $\mathcal{N}_0(\Lambda)$  decays *exponentially* as a function of the number of qubits. Furthermore, we see that even for very small systems (two qubits), the degree of non-Markovianity diminishes very significantly. In Fig. (4-b) we make the same analysis for the relaxation process considering the parameters  $\gamma_0 = 1$  and  $\lambda = 0.1$ . Our findings demonstrate that, unlike in the case of dephasing, the simplified LFS measure  $\mathcal{N}_0(\Lambda)$  might increase for the relaxation process. As a result, comparing Fig. (4-a)

and Fig. (4-b) to Fig. (3-a) and Fig. (3-b), we conclude that  $\mathcal{N}_0(\Lambda)$  might be a misleading quantity for determining the degree of non-Markovianity, despite the fact that it is an easily computable witness of non-Markovianity.

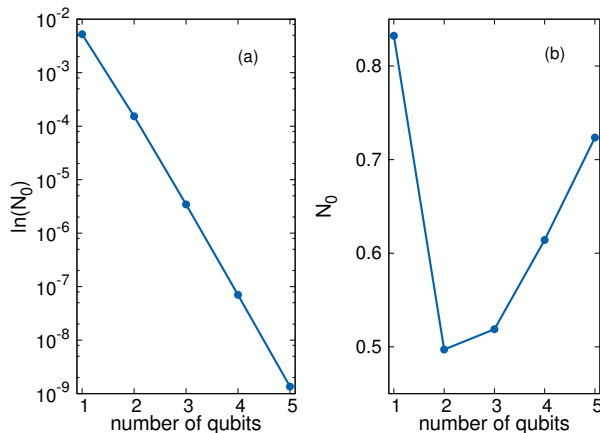


FIG. 4: Logarithm of the non-Markovianity  $\ln(\mathcal{N}_0(\Lambda))$  for the dephasing process (a) with  $s = 3$ ,  $w_c = 1$  and  $\eta = 2$ , and the non-Markovianity  $\mathcal{N}_0(\Lambda)$  for the relaxation process (b) with  $\gamma_0 = 1$  and  $\lambda = 0.1$ , as a function of number of qubits. The graphs are for independent environments.

## VII. SUMMARY

We study the non-Markovianity of independent and common dephasing and relaxation processes for multipartite quantum systems. Considering zero temperature environments, we show that no ancillary system is required to quantify the degree of non-Markovianity of a dynamical quantum process since the LFS measure  $\mathcal{N}(\Lambda)$  can be calculated by the difference of the time derivatives of the system and the environment entropies. This simplification results in an effi-

cient method for calculating the degree of non-Markovianity due to the fact that the Hilbert space, where the maximization is evaluated, does not include an additional ancillary system. We provide an extensive analysis of the non-Markovianity for a single qubit and determine the optimal initial states of the system required for the evaluation of the LFS measure, as a function of the parameters of the environment. We demonstrate that the degree of non-Markovianity might indeed *increase* with the number of the qubits in the system for independent environmental interactions and moreover obtain a lower bound to the LFS measure in this case, namely  $\mathcal{N}(\Lambda^{\otimes n}) \geq n[\mathcal{N}(\Lambda)]$ . Furthermore, we investigate the non-Markovian behavior of two qubits interacting with a common environment and show that the amount of non-Markovianity is amplified as compared to a single qubit for the relaxation process. Finally, we present a comparison between the LFS measure  $\mathcal{N}(\Lambda)$  and its simplified version  $\mathcal{N}_0(\Lambda)$  for independent environments. Our examination clearly suggests that although  $\mathcal{N}_0(\Lambda)$  can be used as a non-Markovianity witness, it may give incorrect results for multipartite systems as a non-Markovianity measure due to the lack of optimization in its evaluation.

## Acknowledgments

We thank Mauro Paternostro, Bogna Bylicka, and L. G. E. Arruda for fruitful discussions. This work is supported by FAPESP and CNPq through the National Institute for Science and Technology of Quantum Information (INCT-IQ) and by the Scientific and Technological Research Council of Turkey (TUBITAK) under Grant 111T232.

*Note added:* While finishing this paper we became aware of a work that is related with the results present in our section (V.B.). There the authors considered the study of a common environment for a different kind of reservoir [11].

- 
- [1] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008); X.-M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A **82**, 042103 (2010); Z. Y. Xu, W. L. Yang, and M. Feng, Phys. Rev. A **81**, 044105 (2010); R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, Phys. Rev. A **84**, 052118 (2011); P. Haikka, S. McEndoo, G. De Chiara, G. M. Palma, and S. Maniscalco, Phys. Rev. A **84**, 031602(R) (2011); B.-H. Liu, L. Li, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, Nat. Phys. **7**, 931 (2011); P. Haikka, J. Goold, S. McEndoo, F. Plastina, and S. Maniscalco, Phys. Rev. A **85**, 060101(R) (2012); A. Chiuri, C. Greganti, L. Mazzola, M. Paternostro, and P. Mataloni, Sci. Rep. **2**, 968 (2012); L. Mazzola, C. A. Rodriguez-Rosario, K. Modi, and M. Paternostro, Phys. Rev. A **86**, 010102(R) (2012).
  - [2] B. Bylicka, D. Chruściński, and S. Maniscalco, arXiv:1301.2585.
  - [3] A. W. Chin, S. F. Huelga, and M. B. Plenio, arXiv:1103.1219.
  - [4] R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **83**, 042321 (2011).
  - [5] M. Žnidarič, C. Pineda, and I. García-Mata, Phys. Rev. Lett. **107**, 080404 (2011); S. Lorenzo, F. Plastina, and M. Paternostro, arXiv:1205.4535.
  - [6] X. M. Lu, X. Wang and C. P. Sun, Phys. Rev. A. **82**, 042103 (2010); Z. Y. Xu, W. L. Yang, and M. Feng, Phys. Rev. A **81**, 044105 (2010); E.-M. Laine, H.-P. Breuer, J. Piilo, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. **108**, 210402 (2012).
  - [7] H.-P. Breuer, E.-M. Laine, J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
  - [8] A. Rivas, S.F. Huelga, and M.B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
  - [9] S. Luo, S. Fu, and H. Song, Phys. Rev. A **86**, 044101 (2012).
  - [10] E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A **81**, 062115 (2010); L. Mazzola, E.-M. Laine, H.-P. Breuer, S. Maniscalco, and J. Piilo, Phys. Rev. A **81**, 062120 (2010); S. C. Hou, X. X.

- Yi, S. X. Yu, and C. H. Oh, Phys. Rev. A **83**, 062115 (2011).  
[11] C. Addis, P. Haikka, S. McEndoo, C. Macchiavello, and S. Maniscalco, arXiv:1302.4259.